

Carrots for dessert

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Abstract

Carrots for dessert is the title of a section of the paper ‘On polynomial-like mappings’, [DH]. In that section Douady and Hubbard define a notion of dyadic carrot fields of the Mandelbrot set and more generally for Mandelbrot like families (for a precise statement see below). They remark that such carrots are small when the dyadic denominator is large, but they do not even try to prove a precise such statement. In this paper we formulate and prove a precise statement of asymptotic shrinking of dyadic Carrot-fields around \mathbf{M} . The same proof carries readily over to show that the dyadic decorations of copies M' of the Mandelbrot set \mathbf{M} inside \mathbf{M} and inside the parabolic Mandelbrot set \mathbf{M}_1 shrink to points when the denominator diverge to ∞ .

Introduction

For $c \in \mathbb{C}$ let $Q_c(z) = z^2 + c$ and let J_c and K_c denote respectively the Julia set and the filled Julia set for Q_c . Denote by \mathbf{M} the Mandelbrot set

$$\mathbf{M} = \{c \in \mathbb{C} \mid Q_c^n(0) \not\rightarrow_{n \rightarrow \infty} \infty\}.$$

Similarly for $B \in \mathbb{C}$ let $g_B(z) = z + 1/z + B$. Then each g_B has a parabolic fixed point at ∞ with multiplier 1 and g_B is conjugate to g_{-B} via $z \mapsto -z$. The parabolic Mandelbrot set \mathbf{M}_1 is the set

$$\mathbf{M}_1 = \{A \in \mathbb{C} \mid \text{either } g_{\sqrt{A}}^n(-1) \not\rightarrow_{n \rightarrow \infty} \infty \text{ or } g_{\sqrt{A}}^n(1) \not\rightarrow_{n \rightarrow \infty} \infty\}. \quad (1)$$

Let T be a closed triangle in the right halfplane \mathbb{H}_+ union $\{0\}$ bounded by lines through the origin and a non horizontal line in such a way that the $i2\pi\mathbb{Z}$ translates are disjoint. Let $\hat{\Delta}_0$ be the image of T under $z \mapsto e^z$. Then by construction $\hat{\Delta}_0$ is simply connected and $Q_0^{-1}(\hat{\Delta}_0)$ has two connected components one, which is a subset of $\hat{\Delta}_0$ and another one $\hat{\Delta}_{1/2}$ containing $-1 = e^{i2\pi/2}$. Define recursively $\hat{\Delta}_{p/2^n}$ as the connected component of $Q_0^{-n}(\hat{\Delta}_0)$ containing $\exp(i2\pi p/2^n)$. The sets $\hat{\Delta}_0$ and $\hat{\Delta}_{p/2^n}$, $0 < p < 2^n$, p odd and $n \in \mathbb{N}$ are disjoint. Together they form a “dyadic Carrot field” $\hat{\Delta}$ of \mathbb{D} :

$$\hat{\Delta} = \hat{\Delta}_0 \cup \bigcup_{n \geq 1} \bigcup_{0 < p < 2^n} \hat{\Delta}_{p/2^n}.$$

The degenerate version of such a carrot field is a “dyadic stick field” defined similarly, but with $T = [1, t]$ for some $t > 1$. We shall in the following denote by carrot field any possibly degenerate carrot field.

Let $\Psi : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \overline{\mathbb{C}} \setminus \mathbf{M}$ denote the Douady-Hubbard uniformizing parameter. That is Ψ is biholomorphic, tangent to the identity at ∞ and its inverse Φ is given by $\Phi(c) = \phi_c(c)$, where ϕ_c denotes the Böttcher-coordinate of Q_c at ∞ . We shall use also the Green’s functions for \mathbf{M} and K_c , i.e. the subharmonic functions $g_{\mathbf{M}}(c) = \log^+ |\phi_c(c)|$ and

$$g_c(z) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \log^+ |Q_c^n(z)|.$$

Moreover we shall use the notation $E_{\mathbf{M}}(h)$ and $E_c(h)$ for the equipotentials for $g_{\mathbf{M}}$ and g_c of level $h \geq 0$. Similarly we shall use the notation $F_{\mathbf{M}}(h)$ and $F_c(h)$ for the closed filled equipotentials of level or height h :

$$F_{\mathbf{M}}(h) = \{c | g_{\mathbf{M}}(c) \leq h\}, \quad F_c(h) = \{z | g_c(z) \leq h\}.$$

The external ray of argument θ for \mathbf{M} or K_c is the field line of $g_{\mathbf{M}}$ or g_c , which is asymptotic to the halfline $\exp(t + i2\pi\theta)$ at ∞ .

By the Douady-Hubbard landing theorem for rational external rays of \mathbf{M} , Ψ has a continuous extension along any external ray of rational argument. In particular along the rays R_{θ} with dyadic arguments $\theta = p/2^n$. By construction each connected component $\hat{\Delta}_{\theta}$ of $\hat{\Delta}$ is contained in some Stolz angle measured from its vertex $e^{i2\pi\theta} \in \mathbb{S}^1$. It thus follows that the radial extension of Ψ along R_{θ} defines a continuous extension of Ψ to $\Psi(\hat{\Delta}_{\theta})$ for each dyadic θ . Hence $\Psi(\hat{\Delta})$ is well defined. Write $\Delta = \Psi(\hat{\Delta})$ and $\Delta_{p/2^n} = \Psi(\hat{\Delta}_{p/2^n})$.

Then the carrot (or stick) decorated Mandelbrot set is $\mathbf{M} \cup \Delta$, where $\Delta = \Psi(\hat{\Delta})$ is any (possibly degenerate) carrot field.

We can also at least partially transport $\hat{\Delta}$ to the dynamical plane of Q_c and thus obtain $\Delta^c = \phi_c^{-1}(\Delta)$, where we use for ϕ_c^{-1} the maximal radial extension. We can then view the carrots Δ of \mathbf{M} as the set of parameters for which c belongs to the corresponding carrot Δ^c of the filled Julia set K_c .

With this terminology the Theorem of shrinking of dyadic carrots of \mathbf{M} is

Theorem 1. *For any (possibly degenerate) dyadic carrot field Δ of \mathbf{M}*

$$\lim_{n \rightarrow \infty} \text{diam}(\Delta_{p/2^n}) = 0.$$

We shall refer to any of the sets $\Delta_{p/2^n}$ as a dyadic carrot of \mathbf{M} .

An easy adaptation of our proof shows that dyadic is not essential. That is if $\hat{\Delta}$ is a carrot field, where instead $\hat{\Delta}_0$ is any finite collection of disjoint (possibly degenerate) triangles attached to periodic orbits for Q_0 and $\hat{\Delta}$ is obtained by iterated pull back as above. Then the corresponding version of Theorem 1 still holds.

Let M' with period k denote a copy of \mathbf{M} inside \mathbf{M} or a copy of \mathbf{M} inside the Parabolic Mandelbrot set \mathbf{M}_1 . Let $\theta \pm$ be the arguments of the pair of

external rays (parabolic external rays if $M' \subset \mathbf{M}_1$) co-landing at the root c'_0 of the principal hyperbolic component H' for M' . We let $\chi_{M'} : M' \rightarrow \mathbf{M}$ denote the Douady-Hubbard straightening map (for a definition see [DH, Chap. II, 1-4]).

Let $I = I(H') = I(M') = [\theta_-, \theta_+]$ be the tuning interval for M' (or equivalently for H') and let $\widehat{\theta}_+ < \widehat{\theta}_- \in I$ be the points such that each of the subintervals $I_0 = [\theta_-, \widehat{\theta}_+]$ and $I_1 = [\widehat{\theta}_-, \theta_+]$ map diffeomorphically onto I under σ^k , where $\sigma(\theta) = 2\theta \bmod 1$. Let $I^{M'}$ denote the corresponding σ^k -invariant Cantor set and let $\kappa = \kappa_{M'} : I^{M'} \rightarrow \Sigma_2$ denote the conjugacy of $\sigma^k : I^{M'} \rightarrow I^{M'}$ to the shift on $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ with $\kappa(\theta_-) = \overline{0}$ and $\kappa(\theta_+) = \overline{1}$. Then the pair of rays with arguments $\widehat{\theta}_{\pm}$ coland at the principal tip $c'_{1/2} = \chi_{M'}^{-1}(\Psi(1/2))$ of M' . The sector $W'_{1/2}$ bounded by these rays and disjoint from M' is called the principal wake of M' and the intersection $\Delta'_{1/2} := \overline{W'_{1/2}} \cap \mathbf{M}$ is called the $1/2$ dyadic decoration of M' . More generally for p odd with binary representation $p = \epsilon_1 \dots \epsilon_n$, $\epsilon_n \neq 0$ the dyadic number $p/2^n$ has two binary representations $0.\epsilon_1 \dots \epsilon_n \overline{0}$ and $0.\epsilon_1 \dots \epsilon_{n-1} 0\overline{1}$. According to the Douady tuning algorithm $\theta_{p/2^n}^- = \kappa^{-1}(\epsilon_1 \dots \epsilon_n \overline{0})$ and $\theta_{p/2^n}^+ = \kappa^{-1}(\epsilon_1 \dots \epsilon_{n-1} 0\overline{1})$ are the two endpoints of a complementary interval of $I^{M'}$. Moreover the corresponding external rays of \mathbf{M} co-land at the relatively dyadic tip $c'_{p/2^n} = \chi_{M'}^{-1}(\Psi(p/2^n))$ of M' and for any parameter $c \in M'$ the corresponding dynamical rays co-land on a point, which is preperiodic to the relative β fixed point. The $p/2^n$ -wake $W'_{p/2^n}$ and the dyadic decoration $\Delta'_{p/2^n} := \overline{W'_{p/2^n}} \cap \mathbf{M}$ of M' are defined similarly as the $1/2$ wake and decoration. Denote by W'_0 the sector bounded by the rays of arguments θ_{\pm} and not containing M' and let $\Delta'_0 = \overline{W'_0} \cap \mathbf{M}$. Note that for each $p/2^n$ the root $c'_{p/2^n}$ of the corresponding wake or limb is the only point of intersection between M' and (the closure of the) wake or limb. Note also that any two wakes are disjoint. As above we write

$$\Delta' = \Delta'_0 \cup \bigcup_{n \geq 1} \bigcup_{0 < p < 2^n} \Delta'_{p/2^n}.$$

(For M' a copy of \mathbf{M} inside \mathbf{M}_1 we use parabolic rays.).

Then

Theorem 2 (Douady-Hubbard, Yoccoz). *For any copy M' of \mathbf{M} in \mathbf{M} :*

$$\mathbf{M} = M' \cup \bigcup_{n \geq 0} \bigcup_{p/2^n \in \mathbb{Q}} \Delta'_{p/2^n}.$$

Proof. The copy M' of \mathbf{M} belongs to the limb $L_{p'/q'}^{\mathbf{M}}$ of the central hyperbolic component H_0 of \mathbf{M} , for some $p', q' \in \mathbb{N}$ with $(p', q') = 1$. Let $c \in M'$ be the center of M' , i.e. $Q_c^k(c) = c$, where k is the period of M' . Let P_n , $n \in \mathbb{N}$ be the level n (p'/q') -Yoccoz puzzle piece containing the critical value c of Q_c and let \mathcal{P}_n denote (p'/q') -Parameter Yoccoz puzzle piece containing the parameter c . Then for each n the map $\Psi \circ \phi_c(z)$ restricted to $\partial P_n \setminus J_c$ extends

to a homeomorphism of ∂P_n onto $\partial \mathcal{P}_n$ preserving argument and potential. Also for each n the closed puzzle piece \overline{P}_n contains the ends from potential 2^{-n} and down of the external rays with arguments in $I^{M'}$. Hence the same holds for the corresponding parameter rays and $\overline{\mathcal{P}}_n$. It follows that any other level n parameter puzzle piece as well as $\mathbf{M} \setminus L_{p'/q'}^{\mathbf{M}}$ is contained in one of the relatively dyadic wakes $\Delta'_{p/2^m}$ of M' . The theorem then follows from Yoccoz parameter puzzle theorem for renormalizable parameters, which states that

$$M' = \bigcap_{n \geq 0} \overline{\mathcal{P}}_n.$$

□

Theorem 3. *For any copy M' of \mathbf{M} in \mathbf{M}_1 :*

$$\mathbf{M}_1 = M' \cup \bigcup_{n \geq 0} \bigcup_{p/2^n \in \mathbb{Q}} \Delta'_{p/2^n}$$

Proof. Completely analogous to the above. □

The Shrinking decorations Theorem for strict copies M' of \mathbf{M} in \mathbf{M} or \mathbf{M}_1 can then be stated as

Theorem 4. *For any strict copy M' of \mathbf{M} in \mathbf{M} or in \mathbf{M}_1*

$$\lim_{n \rightarrow \infty} \text{diam}(\Delta'_{p/2^n}) = 0.$$

The two theorems Theorem 1 and Theorem 4 have very similar proofs, the proof of the first being slightly more complicated. We shall detail the proof of the first and sketch the difference to the proof of the second.

Dzmitry Dudko presents a different and independent proof of the Shrinking decorations Theorem for strict copies M' of \mathbf{M} in \mathbf{M} in [Du]. His statement includes more generally strict copies of the Multibrot set inside the Multibrot set of the same degree. The proof we give here would also easily extend to the Multibrot case.

Proofs

First reduction: Independence on T .

Going back to the initial setting of possibly degenerate carrot fields decorating \mathbf{M} . We shall first show that the proof of Theorem 1 can be reduced to considering only one particular stick-field.

Indeed let T^1 and T^2 be any two possibly degenerate triangles in $\mathbb{H}_+ \cup \{0\}$. and let $\tilde{T}^i = T^i \setminus \{0\}$ for $i = 1, 2$. Then there exists $\delta > 0$ such that \tilde{T}^1 is contained in a hyperbolic δ -neighbourhood of \tilde{T}^2 in \mathbb{H}_+ and vice versa. As $\exp : \mathbb{H}_+ \rightarrow \mathbb{C} \setminus \mathbb{D}$ and $Q_0 : \mathbb{C} \setminus \mathbb{D} \rightarrow \mathbb{C} \setminus \mathbb{D}$ are hyperbolic isometries the same

statement holds for $\widehat{\Delta}_\theta^i$, $i = 1, 2$ and $\theta = p/2^n$ any dyadic. By elementary estimates on hyperbolic metrics, there exists $k = k(\delta) > 1$ such that for any univalent map $\psi : \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C}$, tangent to the identity at infinity, and any dyadic θ

$$\frac{1}{k} \leq \frac{\text{diam}(\psi(\widehat{\Delta}_\theta^1 \setminus \{e^{i2\pi\theta}\}))}{\text{diam}(\psi(\widehat{\Delta}_\theta^2 \setminus \{e^{i2\pi\theta}\}))} \leq k$$

in particular

$$\frac{1}{k} \leq \frac{\text{diam}(\Delta_\theta^1)}{\text{diam}(\Delta_\theta^2)} \leq k$$

where $\text{diam}(\cdot)$ denotes euclidean diameter.

Hence to prove Theorem 1 it suffices to consider a particular stick field, say the field for $t = 1/2$, which is what we shall do.

The toy, but key argument

To set the scene let us however consider first the toy example, where we replace the interval $T = [0, 1/2]$ defining $\widehat{\Delta}_0$ by a compact subset of \mathbb{H} , whose $i2\pi\mathbb{Z}$ translates are disjoint, i.e. whose projection to $\mathbb{C} \setminus \overline{\mathbb{D}}$ does not separate $\overline{\mathbb{D}}$ from ∞ , say $\widehat{\Delta}_0 = \exp([1/4, 1/2])$. This completely trivialises the problem by considerations on the comparison of hyperbolic and euclidean distance similar to above: In this simpler case the set $\widehat{\Delta}_0$ and thus also Δ_0 has finite hyperbolic diameter $diam$ and moreover this bound on the hyperbolic diameter of $\widehat{\Delta}_0$ is an upper bound on the hyperbolic diameter of any of the dyadic carrots $\Delta_{p/2^n} \subset \Delta$. Hence the euclidean diameter of any such dyadic carrot is bounded uniformly from above by a universal constant $k = k(diam)$ times the euclidean distance between $\Delta_{p/2^n}$ and \mathbf{M} . Since the later tends to zero uniformly as $n \rightarrow \infty$ we have in the toy case

$$\limsup_{n \rightarrow \infty} \text{diam}(\Delta_{p/2^n}) = 0.$$

We shall see that, this is effectively what happens, if we restrict our attention to any renormalization copy M' of \mathbf{M} . However the decorations around M' are not the image under a holomorphic map of a union of compact sets all of which are isometric copies of an initial connected component. Hence we need to devise other means of making hyperbolic estimates. To this end we shall use the fact that if $K \subset V' \subset V \subset U$, with U a hyperbolic domain and $\text{mod}(V \setminus \overline{V'}) \geq \delta > 0$, then the hyperbolic diameter of K in U satisfies $\text{diam}_U(K) \leq d(\delta)$. And we shall use the observation by Shishikura, that holomorphic motions can be used to transfer bounds for (locally) persistent annuli in dynamical space to bounds for corresponding annuli in parameter space (see [R]).

With this in mind let us proceed to the proof of Theorem 1. Then as mentioned above Theorem 4 will follow by using the same proof.

Proof of Theorem 1.

We prove the following result :

Proposition 5. *For any sequence $\{\Delta_k = \Delta_{p_k/2^{n_k}}\}_{k \in \mathbb{N}}$, $n_{k+1} > n_k$ of carrots for \mathbf{M} with roots c_k :*

$$c_k \xrightarrow[k \rightarrow \infty]{} c_\infty \implies \text{diam}(\Delta_k) \xrightarrow[k \rightarrow \infty]{} 0.$$

Remark 6. *Theorem 1 is an easy corollary of this proposition by compactness of the Mandelbrot set. The details are left to the reader.*

Setup for the proof of Proposition 5.

Let $\{\Delta_k = \Delta_{p_k/2^{n_k}}\}_{k \in \mathbb{N}}$, $n_{k+1} > n_k$ be an arbitrary but fixed sequence of carrots for \mathbf{M} with roots c_k converging to c_∞ . Then first of all $c_\infty \in \partial \mathbf{M}$.

We shall use the Levin-Yoccoz parameter space inequality and Yoccoz theorem on local connectivity of \mathbf{M} at Yoccoz parameters, i.e. parameters c , for which Q_c is not (infinitely) renormalizable and has all periodic points repelling. For the version of Proposition 5 leading to a proof of Theorem 4 the simpler Yoccoz (rather than Levin-Yoccoz) parameter space inequality suffices, but for Proposition 5 we need the extension due to Levin:

Theorem 7 (The Yoccoz-Levin Dynamical Inequality). *Let H be any hyperbolic component of \mathbf{M} of period k . Let p/q be any non zero reduced rational and let $W_{p/q}^H$ denote the relative p/q wake of H , bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For any $c \in W_{p/q}^H$ let λ denote the multiplier of the repelling k -periodic common landing point α' of the kq periodic rays $R_{\eta_\pm}^c$. Then α' has combinatorial rotation number p/q and λ has a logarithm Λ such that:*

$$|\Lambda - p/q2\pi i| \leq \frac{2k \log 2 \cos \theta}{q} \frac{\pi}{\omega(c)},$$

where $\theta \in]-\pi/2, \pi/2[$ is the argument of $\Lambda - p/q2\pi i$ and $\omega(c)$ is the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\text{Log } \phi_c(c) \in \{z = x + iy | 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [L, TH. 5.1], the general case is similar. For completeness we give a proof in the Appendix, page 17. \square

Proposition 8. *Let H be a period k hyperbolic component of \mathbf{M} with tuning interval $I(H) = [\theta_-, \theta_+]$ and let $p'/2^{m'} \in I(H)$, $1 \leq m' < k$ be the dyadic with the smallest denominator. For any irreducible rational p/q , let $0 < \eta_- < \eta_+ < 1$ be the arguments of the co-landing parameter rays bounding $W_{p/q}^H$ and let $p/2^m \in [\eta_-, \eta_+]$ be the dyadic with the smallest denominator. We have*

$$2^{-kq} \leq \eta_+ - \eta_- \quad \text{and} \quad m = m' + k(q - 2).$$

Proof. As $\eta_- < \eta_+$ are periodic of exact period kq , we have $\eta_+ - \eta_- \geq 1/(2^{kq} - 1) > 2^{-kq}$. For the second inequality let M' denote the copy of \mathbf{M} with H as central hyperbolic component. Let $\theta_- < \theta_+ \in I^{M'}$ denote the arguments of the parameter rays colanding at the root of M' . Let $I = [\theta_-, \theta_+] \supset I_0, I_1, I^{M'} \subset$

$I_0 \cup I_1$ and $\kappa : I^{M'} \rightarrow \Sigma_2$ be as above and write π for the binary projection of Σ_2 onto \mathbb{T} and set $\hat{\kappa} = \pi \circ \kappa$. Then $\tau_{\pm} = \hat{\kappa}(\eta_{\pm})$ are the arguments of the parameter rays co-landing at the root of the wake $W_{p/q}^{H_0}$. It is well known that the intervals $\sigma^j([\tau_-, \tau_+])$, $0 \leq j < q$ are interiorly disjoint and injective images. Moreover $0 \in \sigma^{(q-1)}([\tau_-, \tau_+])$ and thus $1/2 \in \sigma^{(q-2)}([\tau_-, \tau_+])$. Consequently $\sigma^{k(q-2)}$ maps $[\eta_-, \eta_+]$ injectively into I . Moreover $I \supset \sigma^{k(q-2)}([\eta_-, \eta_+]) \supset (I \setminus (I_0 \cup I_1))$. Let $p'/2^{m'} \in I$ be the dyadic with smallest denominator then $p'/2^{m'} \in I \setminus (I_0 \cup I_1)$ and $1 \leq m' \leq k$. Thus $m = m' + k(q-2)$. \square

Corollary 9. *For any $c \in W_{p/q}^H \cap (\mathbf{M} \cup \Delta)$ the angle of vision $\omega(c)$ of $i2\pi[\eta_-, \eta_+]$ from $\text{Log}(\phi_c(c))$ is bounded from below by*

$$\arctan(2\pi 2^{m'-2k}).$$

Proof. The angle is bounded from below by the angle obtained, when c belongs to one of the two bounding rays of $W_{p/q}^H$:

$$\begin{aligned} \arctan(2\pi(\eta_+ - \eta_-)/\log|\phi_c(c)|) &\geq \arctan(2\pi 2^{-kq}/2^{-(m'+k(q-2))}) \\ &= \arctan(2\pi 2^{m'-2k}). \end{aligned}$$

\square

Theorem 10 (The Yoccoz-Levin Parameter Inequality). *For any hyperbolic component H of \mathbf{M} there exists a constant $C = C_H > 0$ such that for any relative p/q wake $W_{p/q}^H$*

$$\text{diam}(W_{p/q}^H \cap (M \cup \Delta)) \leq \frac{C}{q}.$$

Proof. For $h = 0$ i.e. for the limbs $\mathbf{M} \cap W_{p/q}^H$, this is essentially proved by Hubbard in [H], except that he confuses the direction of the square root from primitive hyperbolic components and obtains an inequality with C/\sqrt{q} instead of C/q in the primitive case. Whereas the bounds actually gives C/q^2 asymptotically when p/q tend to 0 or 1. For the extension we use the Levin-Yoccoz dynamical inequality above instead of the Yoccoz inequality. By Corollary 9 the angle $\omega(c)$ for $c \in W_{p/q}^H \cap (M \cup \Delta)$ is bounded from below by the angle $\omega_H = \arctan(2\pi 2^{m'-2k})$. The argument is then identical to the argument in Hubbards paper [H], except using the Levin-Yoccoz dynamical inequality with the fixed value ω_H . Thus asymptotically for q large we can take

$$C_H = \frac{\pi}{\omega_H} C_H^{\text{Yoccoz}}$$

where C_H^{Yoccoz} is the corresponding asymptotic value for Yoccoz parameter inequality. \square

Let H_0 denote the central hyperbolic component of \mathbf{M} and for p/q an irreducible rational let $W_{p/q}^{H_0}$ denote the p/q wake of H_0 and $L_{p/q}^{H_0} = W_{p/q}^{H_0} \cap \mathbf{M}$ the p/q limb.

Theorem 11 (Yoccoz). *For any p/q and any $c \in L_{p/q}^{H_0}$ there are two possibilities; either c is not renormalizable and the p/q parameter puzzle pieces containing c nests down to c , or c is at least once renormalizable, say with first renormalization period k and there is a first level n such that for the dynamical puzzle pieces $P_n =: U$ and $P_{n+k} =: U'$, $Q_c^k : U \rightarrow U'$ is quadratic like with connected filled-in Julia set. (fattening U and U' if $k = q$.)*

For a proof see [H].

The second reduction : reduction to renormalizable c_∞ .

Let us first apply the Yoccoz-Levin parameter inequality. This gives a constant $C > 0$ such that for all p/q

$$\text{diam}(W_{p/q}^{H_0} \cap (\mathbf{M} \cup \Delta)) \leq \frac{C}{q}.$$

The sequence $\{\Delta_k\}$ of carrots is included in a sequence of Wakes $W_{p_k/q_k}^{H_0}$. Hence it follows that if q_k tends to ∞ the diameter of Δ_k tends to 0. Else, there is $Q_1 \in \mathbb{N}$ such that

$$c_\infty \in \partial(L_{p/q}^{H_0})$$

for some p/q with $q \leq Q_1$ and moreover the carrots Δ_k are eventually contained inside $W_{p/q}^{H_0}$, because there are finitely many wakes $W_{p/q}^{H_0}$ with $q \leq Q_1$ and they are strongly separated.

Secondly we apply Yoccoz parameter puzzles theorem, Theorem 11. For any p/q the corresponding rotation orbit $0 < \theta_0 < \dots < \theta_{q-1}$ is disjoint from the set of dyadic arguments. Thus for any p/q the graph defining the associated p/q puzzle for $L_{p/q}^{H_0}$ is disjoint from $\Psi(\widehat{\Delta})$. Therefore, there exists an increasing sequence n_k such that for $k \geq k_0$ the carrots $\Delta_k \subset \overline{\mathcal{P}}_{n_k} \ni c_\infty$ (the Parameter Puzzle Piece).

Hence by Yoccoz Theorem 11 either the diameter tends to 0 or the limiting parameter c_∞ is renormalizable, that is $c_\infty \in M'$ for some period k first renormalization copy M' of \mathbf{M} in $L_{p/q}^{H_0}$, where $q \leq Q_1$ and $q \leq k$.

The third reduction : reduction to the toy example.

As above let θ_\pm denote the arguments of the external rays of \mathbf{M} co-landing at the root c'_0 of M' . Let Λ denote the parameter disk whose closure contains M' and which is bounded by the segments of the rays $R_{\theta_\pm}^{\mathbf{M}}$ with potential up to and including 2 union a connecting subarc of the level 2 equipotential $E_{\mathbf{M}}(2) := \Psi(C(0, e^2))$, where $C(0, e^2) = \{z \mid |z| = e^2\}$.

We need the following result on \mathbf{M} presumably due to Douady, Hubbard and Lavaurs.

Theorem 12. *Let $0 < \eta_- < \eta_+ < 1$ be rationals for which the parameter rays $R_{\eta_{\pm}}^{\mathbf{M}}$ co-land at some point $c_0 \in \mathbf{M}$ and let $W_{\eta_-, \eta_+}^{\mathbf{M}}$ denote the parameter sector bounded by $R_{\eta_-}^{\mathbf{M}} \cup \{c_0\} \cup R_{\eta_+}^{\mathbf{M}}$ and not containing H_0 . Then the forward orbits of η_{\pm} do not enter the interval $]\eta_-, \eta_+[$ and for any $c \in W_{\eta_-, \eta_+}^{\mathbf{M}}$ the pair of dynamical rays $R_{\eta_{\pm}}^c$ move homomorphically with c , co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{k'+l}(z(c)) = Q_c^l(z(c))$, where $l \geq 0$ is the common preperiod of η_{\pm} and $k' > 0$ divides the common period $k > 0$ of $\sigma^l(\eta_{\pm})$ and the set $R_{\eta_-}^c \cup \{z(c)\} \cup R_{\eta_+}^c$ bounds a sector W^c containing c , but not 0.*

This theorem is at least folklore. But because we do not have a precise reference, we have for completeness provided a proof in the Appendix, on page 19.

Thus for any $c \in \Lambda$ the dynamical rays $R_{\theta_{\pm}}^c$ co-land at a repelling k -periodic point β'_c and the rays $R_{\theta_{\pm}}^c$ co-land at the Q_c^k -preimage $co\beta'_c$ of β'_c all of which moves holomorphically with $c \in \Lambda$. Moreover the set

$$\bigcup_{j=0}^{k-1} Q_c^{-j}(R_{\theta_-}^c \cup \{\beta'_c\} \cup R_{\theta_+}^c)$$

does not enter the sector W_{θ_-, θ_+}^c bounded by the closure of the colanding pair of rays $R_{\theta_-}^c$ and $R_{\theta_+}^c$ and containing c . Hence

$$E_c(1) \bigcup_{j=0}^k Q_c^{-j}(R_{\theta_-}^c \cup \{\beta'_c\} \cup R_{\theta_+}^c)$$

moves holomorphically with $c \in \Lambda$. Similarly to the relatively dyadic wakes of M' we define the relatively dyadic wakes W_0^c as the open set not containing c and bounded by the closure of the rays $R_{\theta_{\pm}}^c$ and $W_{1/2}^c = W_{\widehat{\theta}_+, \widehat{\theta}_-}^c$ as the open set bounded by the closure of the rays $R_{\theta_{\pm}}^c$ and disjoint from W_0^c . For any $c \in \Lambda$ there is a renormalization, a quadratic-like restriction of Q_c^k for which the filled-in Julia set $K'_c \subset K_c$ consists of the points in the filled-in Julia set of K_c , whose orbits never enters the relatively dyadic wakes W_0^c and $W_{1/2}^c$ (see also (2) below).

The key point in the proof of Proposition 1 is that all of the dyadic carrots $\Delta_{p/2^n}$ are disjoint from M' , because their root points $\Psi(p/2^n)$ are disjoint from M' . And if such a carrot intersects Λ , then it is entirely contained in Λ and its dynamical counter part in the dynamical planes of Q_c is either contained in $W_{1/2}^c$ or has a univalent forward image, which is. In order to prove the theorem we shall wrap the dynamical counter part of each dyadic carrot inside the relatively dyadic wake $W_{1/2}^c$ in an annulus in $W_{1/2}^c$ moving holomorphically with $c \in \Lambda$ and of modulus bounded uniformly from below.

To do this we shall follow slightly different paths according to whether M' is a primitive copy or the satellite copy $M_{p/q}$ with root on the cardioid. We start with the primitive case and afterwards indicate the changes which make the proof in the satellite case.

The Primitive Case

Suppose $M' \subset L_{p/q}$ is a primitive copy of \mathbf{M} . Let δ_0^c denote the subarc of

$$R_{\theta_-} \cup \{\beta'_c\} \cup R_{\theta_+} = \overline{R_{\theta_-} \cup R_{\theta_+}}$$

consisting of points with potential up to and including 1. Similarly let $\delta_{1/2}^c$ denote the subarc of $\overline{R_{\hat{\theta}_-} \cup R_{\hat{\theta}_+}}$ up to and including potential 1. Define similarly the parameter arcs $\delta_0^{M'}$ and $\delta_{1/2}^{M'}$. Let $c \in M'$ be arbitrary and let $P = P_n^c$ denote the p/q puzzle piece of level n containing c given by Theorem 11. Let $\eta^- < \eta^+$ denote the (rational) arguments of the co-landing pair of external rays for Q_c , which are on the boundary of P and which separates c from 0. Then the parameter rays $R_{\eta^\pm}^{\mathbf{M}}$ co-land at some point in \mathbf{M} . Denote by Λ^P the parameter disk which contains Λ , and which is bounded by a subarc of $\overline{R_{\eta_-}^{\mathbf{M}} \cup R_{\eta_+}^{\mathbf{M}}}$ union a subarc of $E^{\mathbf{M}}(2)$. Then by Theorem 12 the dynamical rays $R_{\eta^\pm}^c$ co-land for every $c \in \Lambda^P$ and the arc $\overline{R_{\eta_-}^c \cup R_{\eta_+}^c}$ moves holomorphically with $c \in \Lambda^P$. Denote by γ_0^c the subarc of $\overline{R_{\eta_-}^c \cup R_{\eta_+}^c}$ of potential up to and including 1 and for $c \in \Lambda$ let U_0^c denote the disk not containing 0 and which is bounded by γ_0^c and a subarc of the equipotential $E^c(1)$. Then ∂U_0^c moves holomorphically over Λ^P . Write $\Lambda_0^P := \Lambda^P$ and $\Lambda_1^P = \Lambda^P \cap F^{\mathbf{M}}(1)$. Moreover for $c \in \Lambda_1^P$ let U_1^c denote the connected component of $Q_c^{-k}(U_0^c)$ containing the ends $(R_{\theta_-}^c \cup R_{\theta_+}^c) \cap F^c(2^{-k})$. Then for $c \in \Lambda_1^P$ the restriction

$$f_c := Q_c^k : U_1^c \longrightarrow U_0^c \quad (2)$$

is quadratic like, ∂U_1^c moves holomorphically with c and the filled Julia set K'_c is connected, if and only if $c \in M'$. Let $\omega_c \in U_1^c$ denote the unique critical point of f_c , so that $f_c(\omega_c) = Q_c^k(\omega_c) = c$. Notice that Λ_1^P is precisely the set of parameters for which $c \in U_0^c$, in fact $c \in \partial U_0^c$ if and only if $c \in \partial \Lambda_1^P$. For later use we define $U_n^c = f_c^{-n}(U_0^c)$, which may or may not be connected for $n > 1$.

Write $\Lambda_0 = \Lambda \subset \Lambda_0^P$ and $\Lambda_1 = \Lambda \cap \Lambda_1^P$. For $c \in \Lambda_0$ let V_0^c be the connected component of $U_0^c \setminus \delta_0^c$ containing ω_c let Ξ_0^c denote the other connected component. Define recursively $V_n^c = f_c^{-n}(V_0^c)$ and $\Xi_n^c = f_c^{-n}(\Xi_0^c) \cap \Xi_0^c$ (see also Fig. 1). Then the restriction $f_c : V_{n+1}^c \longrightarrow V_n^c$ is a 2 : 1 branched covering, whereas $f_c : \Xi_{n+1}^c \longrightarrow \Xi_n^c$ is an isomorphism.

Let γ_1^c denote the extension to potential level 1 of $Q_c^{-k}(\gamma_0^c) \cap \partial \Xi_0^c$ and let $B_0^c \subset \Xi_0^c$ denote the quadrilateral bounded by γ_0^c , γ_1^c and subarcs of $E^c(1)$. Define recursively the univalently iterated preimages $B_{n+1}^c = f_c^{-1}(B_n^c) \cap \Xi_n^c$.

For each $n \geq 1$ let $\hat{\Xi}_n^c$ denote the “other” connected component of $f_c^{-1}(\Xi_{n-1}^c)$, having a boundary arc in $\delta_{1/2}^c$. For each $n \geq 1$ let $\hat{B}_n^c \subset \hat{\Xi}_n^c$ denote the “twin” of B_n^c , i.e. the connected component of $f_c^{-1}(B_{n-1}^c) \cap \hat{\Xi}_n^c$. Let $\hat{\gamma}_1 = Q_c^{-k}(\gamma_0^c) \cap \partial \hat{\Xi}_1^c$ extended to equipotential level 3/4 and let Ω^c denote the open disk bounded by $\hat{\gamma}_1^c$ and the subarc of $E^c(3/4) \cap U_0^c$ connecting the endpoints of $\hat{\gamma}_1^c$. Let $D^c \subset V_0^c$, denote the disc bounded by $\delta_{1/2}^c$ union the subarc of $E^c(1)$ connecting the endpoints of $\delta_{1/2}^c$. Then by construction each of the sets Ω^c and \hat{B}_n^c , $n \geq 1$ are relatively compact in D^c . In fact

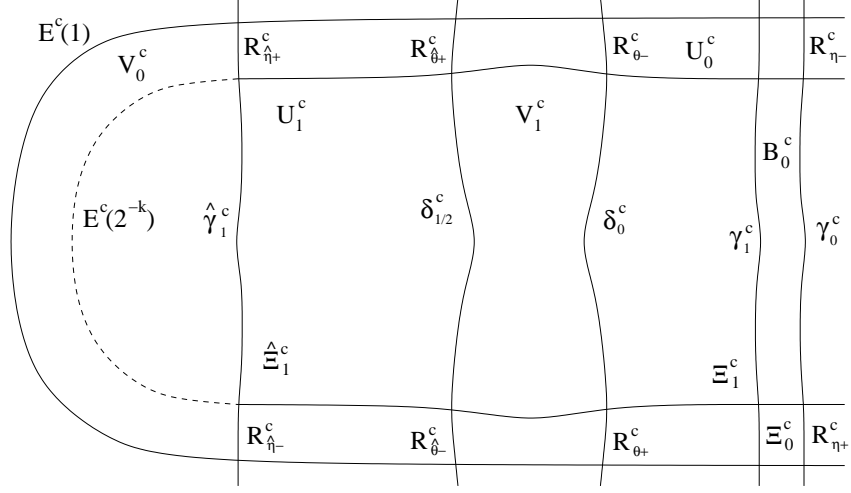


Figure 1: The disks U_0^c , U_1^c , and $\hat{\Xi}_1^c$. The set U_0^c is the disk insided $E^c(1)$ and to the left of γ_0^c . The set U_1^c is the disk inside $E^c(2^{-k})$ and bounded to the right and left by γ_1^c and $\hat{\gamma}_1^c$ respectively. The arc δ_0^c separates the two subdisks V_0^c and Ξ_0^c of U_0^c , V_0^c to the left and Ξ_0^c to the right of δ_0^c . The subsets $\hat{\Xi}_1^c$, V_1^c and Ξ_1^c of U_1^c are to the left of $\delta_{1/2}^c$, between $\delta_{1/2}^c$ and δ_0^c and to the right of δ_0^c respectively.

Lemma 13. *For every $c \in \Lambda$ there exists $m = m(c) > 0$ such that*

$$\text{mod}(D^c \setminus \Omega^c) \geq m, \quad \text{and} \quad \forall n \geq 1 \quad \text{mod}(D^c \setminus \overline{\hat{B}_n^c}) \geq m$$

so that

$$\forall n \geq 1 \quad \text{mod}(\Xi_0^c \setminus \overline{B_n^c}) \geq m.$$

Proof. The restriction $f_c : \hat{\Xi}_1^c \rightarrow \Xi_0^c$ is biholomorphic so that for all $n \geq 1$

$$\begin{aligned} \text{mod}(D^c \setminus \overline{\hat{B}_{n+1}^c}) &\geq \text{mod}(\hat{\Xi}_1^c \setminus \overline{\hat{B}_{n+1}^c}) = \text{mod}(\Xi_0^c \setminus \overline{B_n^c}) \\ &\geq \text{mod}(\Xi_{n-1}^c \setminus \overline{B_n^c}) = \text{mod}(\Xi_0^c \setminus \overline{B_1^c}) \end{aligned}$$

Thus we may define

$$m(c) = \min\{\text{mod}(D^c \setminus \Omega^c), \text{mod}(D^c \setminus \overline{\hat{B}_1^c}), \text{mod}(\Xi_0^c \setminus \overline{B_1^c})\}.$$

□

Moreover again by construction the graph

$$G^c = \partial D^c \cup \partial \Omega^c \cup \bigcup_{n=1}^{\infty} (\partial \hat{\Xi}_n^c \cup \partial \hat{B}_n^c) \cup \bigcup_{n=0}^{\infty} (\partial \Xi_n^c \cup \partial B_n^c) \cup \partial U_0^c$$

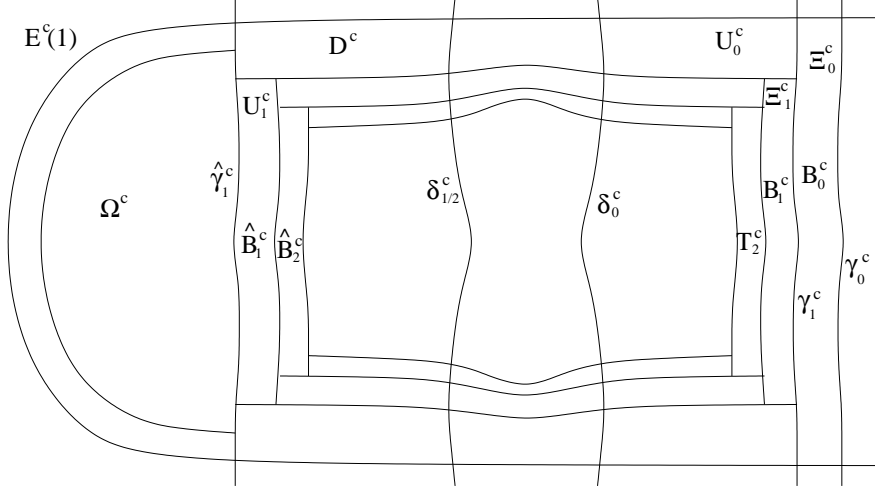


Figure 2: The decomposition of the disk D^c , which is bounded by the equipotential $E^c(1)$ and the arc $\delta_{1/2}^c$. The disk Ξ_0^c is the subset of U_0^c to the right of δ_0^c . The disk $\widehat{\Xi}_1^c$ (not labelled) is the subset of U_1^c to the left of $\delta_{1/2}^c$. The disc Ω^c is to the left of $\widehat{\gamma}_1^c$ and inside the equipotential $E^c(3/4)$ (not labelled).

moves holomorphically with $c \in \Lambda_0$ via the Böttcher-coordinates at ∞ , and does not intersect any of the dynamical dyadic carrots $\Delta_{p/2^m}^c$. The later because all such carrots are at dyadic angles and only Δ_0^c extends further than potential $1/2$. The holomorphic motion part because, $Q_c^n(G^c)$ does not meet the critical point 0 for any $n \geq 0$ or $c \in \Lambda$: When the parameter $c \in \Lambda_0 = \Lambda$ the critical value c does not belong to $\overline{\Xi}_0^c$. Hence the Böttcher-coordinate is defined and depends holomorphically on c , on the dense subset $(\partial\Xi_0^c \cup \partial B_0^c) \setminus K_c$ of $(\partial\Xi_0^c \cup \partial B_0^c)$, so that the later moves holomorphically with c . Secondly f_c depends holomorphically on c and its critical value c again still does not belong to $\overline{\Xi}_0^c$. Hence the iterated univalent preimages of $(\partial\Xi_0^c \cup \partial B_0^c)$ inside Ξ_0^c and $\widehat{\Xi}_1^c$ depend also holomorphically on c . This takes care of

$$\bigcup_{n=1}^{\infty} (\partial\widehat{\Xi}_n^c \cup \partial\widehat{B}_n^c) \cup \bigcup_{n=0}^{\infty} (\partial\Xi_n^c \cup \partial B_n^c).$$

Finally ∂D^c moves holomorphically with c , because $\delta_{1/2}^c$ does and $\partial\Omega^c$ moves holomorphically with c , because $\widehat{\gamma}_1^c$ does.

No dynamical plane dyadic carrot $\Delta_{p/2^n}^c$ intersects G^c . Hence any such dyadic carrot in the relative dyadic wake $W_{1/2}^c$ of the filled Julia set K'_c for $f_c : U_1^c \rightarrow U_0^c$ is contained in one of the sets Ω^c or \widehat{B}_n^c for some $n \geq 1$. Such a carrot is thus wrapped by an annulus of modulus at least $m(c) > 0$, contained in D^c and thus disjoint from K'_c .

We shall use an argument to transfer bounds on moduli of dynamical annuli

to bounds on moduli of corresponding parameter annuli. This argument was pioneered by Shishikura (see also [R]).

Define $\Lambda_1 = \Lambda \cap F(1)$ and fix as basepoint $c_b \in \Lambda_1 \subset \Lambda_0$ the center of the central hyperbolic component of M' . Note that on $F(2) \supset \Lambda_0$ the Böttcher-coordinate at ∞ defines a holomorphic motion of the set $\overline{\mathbb{C}} \setminus F^{c_b}(1)$ extending the Böttcher-motion of G^{c_b} . By Ślodkowski's extension theorem there exists a global holomorphic motion $H : \Lambda_0 \times \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ over Λ_0 with base point c_b and extending the Böttcher-motion of the graph G^{c_b} union $\overline{\mathbb{C}} \setminus F^{c_b}(1)$, in particular we obtain holomorphic motions of $\overline{U_0^{c_b}} \supset \overline{D^{c_b}}$. As usual for $c \in \Lambda$ write $H_c(\cdot) = H(c, \cdot)$, then each map $H_c : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ is a quasi-conformal homeomorphism with a dilatation bounded uniformly above by $\log d_\Lambda(c, c_b)$, where $d_\Lambda(\cdot, \cdot)$ denotes hyperbolic distance in $\Lambda = \Lambda_0$.

Define similarly to D^c the parameter disk $D_\mathbb{C}^{M'} \Lambda_1 \setminus M'$ as the disc bounded by $\delta_{1/2}^{M'}$ union the subarc of $E^{M'}(1)$ connecting the endpoints of $\delta_{1/2}^{M'}$. Then $D^{M'}$ is relatively compact in Λ_0 . Let $\chi : \Lambda_0 \rightarrow \overline{\mathbb{C}}$ be the map $\chi(c) = H_c^{-1}(c)$. Then χ is locally quasi-regular with dilatation $K(c)$ bounded by $\log d_\Lambda(c, c_b)$. By construction the restriction $\chi : \partial D_\mathbb{C}^{M'} \rightarrow \partial D^{c_b}$ is a homeomorphism. Hence the restriction $\chi : \overline{D^{M'}} \rightarrow \overline{D^{c_b}}$ is the restriction of a quasi-conformal homeomorphism with dilatation bounded by $K = K_{1/2} = \max\{\log d_\Lambda(c, c_b) | c \in \overline{D^{M'}}\}$.

Define $\Omega^{M'} = \chi^{-1}(\Omega^{c_b}) \subset D^{M'}$ and $\hat{B}_n^{M'} = \chi^{-1}(\hat{B}_n^{c_b}) \subset D^{M'}$, $n \geq 1$. Then any dyadic carrot $\Delta_{p/2^n} \subset W'_{1/2}$ is contained in one of the disks $\Omega^{M'}$ or $\hat{B}_n^{M'}$ and is thus wrapped in an annulus with a modulus bounded from below by $K_{1/2} \cdot m(c_b)$ according to Lemma 13.

Rename $D^c =: D_{1/2}^c$, $D^{M'} =: D_{1/2}^{M'}$ and $\chi =: \chi_{1/2}$. We have proved that any dyadic carrot $\Delta_{p/2^n}$ in the relative $1/2$ wake $W'_{1/2}$ of M' is wrapped in an annulus of modulus uniformly bounded from below and contained in the disk $D_{1/2}^{M'}$, which is disjoint from M' . Moreover the annuli are q-c images of corresponding annuli in the dynamical plane of Q_{c_b} . We shall prove by induction the similar statements for any other relative dyadic wake $W'_{r/2^s}$ of M' . The only difference is that the bounds on the dilatation of the q-c homeomorphisms $\chi_{r/2^s}$ and hence on the moduli of annuli in the $W'_{r/2^s}$ wake depends on $r/2^s$. As a remedy for this we shall apply the Levin-Yoccoz parameter inequality once more. Here follow the details.

Recall that V_0^c is the connected component of $U_0^c \setminus \delta_0^c$ containing the critical point ω_c and $V_n^c = f_c^{-n}(V_0^c)$. We shall need also the extension $\tilde{V}_1^c = V_0^c \setminus \overline{D^c}$ of V_1^c and its iterated preimages $\tilde{V}_n^c = f_c^{-n}(\tilde{V}_1^c)$. Define parameter disks Λ_s , $s \geq 1$ by

$$\Lambda_s = \{c | c \in \tilde{V}_{s-1}^c\}.$$

Evidently $\Lambda_s \supset \Lambda_{s+1}$. Note that the condition $c \in \tilde{V}_s^c$ is equivalent to $f_c^s(\omega_c) \in \tilde{V}_1^c$. Rename $G^c =: G_1^c$ and define recursively, $G_{s+1}^c = f_c^{-1}(G_s^c) \cup G_1^c$ for $s \geq 1$. Then as noted above G_1^c moves holomorphically in $\Lambda_0 \supset \Lambda_1$ and we shall prove as part of the induction on $s \geq 2$, that for $c \in \Lambda_s$ the critical value $c \notin G_{s-1}^c$, so that G_s^c moves holomorphically over Λ_s .

For $s = 2$ notice that, by the above $c \in \overline{D_{1/2}^c}$ if and only if $c \in \overline{D_{1/2}^{M'}}$. Thus $c \notin G_1^c$ for any $c \in \Lambda_2$, so that G_2^c moves holomorphically with $c \in \Lambda_2$. For $c \in \Lambda_2$ let $D_{r/2^2}^c$ for $r = 1, 3$ denote the connected components of $f_c^{-1}(D_{1/2}^c)$ containing the $r/2^2$ dyadic carrots and define $D_{r/2^2}^{M'} \subset \Lambda_2$ as the parameter disks bounded by the corresponding parameter ray segments and equipotential level.

Rename the previous holomorphic motion H to H_1 and let H_2 denote the restriction of H_1 to $\Lambda_2 \times (\overline{\mathbb{C}} \setminus \widehat{V}_1^{c_b})$. Extend H_2 to a motion including $\widehat{V}_1^{c_b} \setminus V^{c_b}_1$ using the Böttcher-motion and extend H_2 further to $\Lambda_2 \times f_{c_b}^{-1}(\overline{D_{1/2}^{c_b}})$ by $f_c^{-1} \circ H_1(c, f_{c_b}(z))$, where the inverse branches are taken so as to map $D_{r/2^2}^{c_b}$ quasi-conformally onto $D_{r/2^2}^c$. Finally use Slodkowski's extension theorem to extend this holomorphic motion to a holomorphic motion of $\overline{\mathbb{C}}$ over the disk Λ_2 (i.e. extend the motion by a motion of $V^{c_b}_2$). By the same argument as above the map $\chi_2 : \Lambda_2 \rightarrow \widehat{V}_1^{c_b}$ given by $\chi_2(c) = (H_2)_c^{-1}(c)$ is a locally quasi regular map. Again by construction $\chi_2 : \partial D_{r/2^2}^{M'} \rightarrow \partial D_{r/2^2}^{c_b}$ are homeomorphisms so that the restrictions $\chi_2 : D_{r/2^2}^{M'} \rightarrow \partial D_{r/2^2}^{c_b}$ are quasi-conformal. However on the sets $D_{r/2^2}^{M'}$ the holomorphic motion H_2 is a conjugacy between the holomorphic maps f_{c_b} and f_c . Hence the dilatation of $(H_2)_c$ at z equals that of $(H_1)_c$ at $f_{c_b}(z)$. Hence again the dilatation of χ_2 on $D_{r/2^2}^{M'}$ is again bounded by bound given by $K_{r/2^s} = \max\{\log d_\Lambda(c, c_b) | c \in \overline{D_{r/2^2}^{M'}}\}$. Arguing as in the initial case corresponding to $s = 1$ completes the case $s = 2$. The inductive step is completely similar and is left to the reader.

Let H' denote the central hyperbolic component of M' . Then for k sufficiently large c_k belongs to the p'_k/q'_k limb $L_{p'_k/q'_k}^{H'}$ of H' . Applying the Yoccoz-Levin parameter inequality Theorem 10 to H' we find that the diameter of $L_{p'/q'}^{H'}$ union its attached dyadic carrots is bounded uniformly by C/q' for some constant $C = C_{H'}$. Thus it suffices to consider the case $q'_k \leq Q'$ for some integer Q' . As the containing wakes $W_{p'/q'}^{H'}$, $q' \leq Q'$ are strongly separated we can further assume that $p'_k/q'_k = p'/q'$ for k large. The set $W_{p'/q'}^{H'} \cap \Lambda_1$ is relatively compact in Λ so that

$$\sup\{\log d_\Lambda(c, c_b) | c \in W_{p'/q'}^{H'} \cap \Lambda_1\} = K = K_{p'/q'}^{H'} < \infty.$$

Hence the dyadic carrots Δ_k either has a diameter which a priori tends to zero or such carrots are separated from M' by an annulus in $\Lambda \setminus M'$ of modulus at least $m(c_b)/K$. And in the latter case their diameters are forced to converge to zero a posteriori. Because the roots $c_k \in \Delta_k$ converge to $c_\infty \in M'$,

This completes the proof that if $\Lambda \ni c_k \rightarrow c_\infty \in M'$, then the diameter of Δ_k converge to zero. For the case $c_k \in \Lambda_P \setminus \Lambda_0$ we necessarily have $c_\infty = c_r$, where c_r denotes the root of M' . To prove that the diameter of Δ_k converge to zero also in this case let

$$\Lambda_n^P = \{c \in \Lambda_P | c \in U_n^c\}.$$

For any $c \in \Lambda = \Lambda_0$ the sets $\partial \Xi_n^c$, $n \geq 0$ move holomorphically with c . Define $A_n^c = \Xi_n^c \setminus \Xi_{n+1}^c$, then the A_n^c are quadrilaterals with a-sides the boundary arcs

$\partial A_n^c \cap R_{\theta_-}^c$ and $\partial A_n^c \cap R_{\theta_+}^c$. Moreover ∂A_n^c even move holomorphically with $c \in (\Lambda_0 \cup \Lambda_n^P)$. Let

$$A_n^{M'} = \{c \in \Lambda_n^P | c \in A_n^c\}$$

denote the corresponding parameter quadrilaterals. Then the root c_r of M' belong to Λ_n^P , for all n . Choose by Slodkowski's extension theorem a holomorphic motion

$$H^0 : \Lambda_0^P \times A_0^{c_r} \longrightarrow \mathbb{C}$$

over Λ_0^P with base point c_r of the quadrilateral $A_0^{c_r}$ extending the Böttcher motion of its boundary.

For $c \in \Lambda_n^P$ the restriction $f_c^n : A_n^c \longrightarrow A_n^0$ is biholomorphic. Hence we may lift the motion H^0 to a holomorphic motion

$$H^n : \Lambda_n^P \times A_n^{c_r} \longrightarrow \mathbb{C}.$$

As with the annuli above define quasi-conformal homeomorphisms

$$\rho_n : A_n^{M'} \longrightarrow A_n^{c_r}$$

by $\rho_n(c) = (H_c^n)^{-1}(c)$. Then as above these have q.-c. distortion bounded by the distortion of the q.-c. homeomorphisms $H_c^0(\cdot)$, $c \in A_n^{M'}$. That is bounded by

$$K = \sup\{\log d_{\Lambda_0^P}(c, c_r) | c \in A_n^{M'}\}$$

which is uniformly bounded, because $A_n^{M'} \subset \Lambda_1^P \subset \Lambda_0^P$. Thus all the quadrilaterals $A_n^{M'}$ have modulus bounded uniformly from below by $\text{mod}(A_0^{c_r})/K$. Moreover the a-sides of these quadrilaterals are all contained in the two rays $R_{\theta_-}^{M'}$ and $R_{\theta_+}^{M'}$ co-landing at c_r . By the Grötzsch-inequality for annuli the euclidean diameter of $A_n^{M'}$ tend to zero and the closures converge to c_r . By construction no dyadic carrot intersects the boundary of any of the $A_n^{M'}$. Thus also in this case the diameter of Δ_k converge to zero as $k \rightarrow \infty$. This completes the proof in the case c_∞ belongs to a primitive first renormalization copy.

The satellite case

In the complementary satellite case $M' = M_{p/q}$ with central hyperbolic component $H_{p/q}$ attached at internal argument $\exp(i2\pi p/q)$ from the central hyperbolic component H_0 of \mathbf{M} . Let as above $\theta_- < \theta_+$ be the arguments of the parameter rays co-landing at the root and bounding the wake $W_{p/q}^{H_0}$. Recall that $c_\infty \in M_{p/q}$ is the limiting parameter of the roots of dyadic carrots and that these dyadic carrots are eventually contained in $W_{p/q}^{H_0}$.

We apply the Yoccoz-Levin parameter inequality Corollary 10 similarly as we have done twice above. This reduces the problem to the case where c_∞ belongs to a relative p'/q' -limb $L_{p'/q'}^{H_{p/q}}$ of $H_{p/q}$ for some $p'/q' \neq 0/1$ and the dyadic carrots Δ_k are subsets of the corresponding wake $W_{p'/q'}^{H_{p/q}}$ for large k . Denote

by $\tau_- < \tau_+$ the arguments of the parameter rays bounding $W_{p'/q'}^{H_{p/q}}$ and define $\Lambda = \Lambda_0 = W_{p'/q'}^{H_{p/q}} \cap F^{\mathbf{M}}(2)$. Let $I^{M'} = I^{M_{p/q}}$ denote the Cantor set of arguments of parameter rays accumulating M' as given by the Douady tuning algorithm. Then $\tau_{\pm} \in I^{M'}$ and each has a unique preimage $\hat{\tau}_{\pm} = \sigma^{-q}(\tau_{\pm}) \cap I^{M'}$ different from itself. For $c \in \Lambda$ let U_0^c denote the disk containing the fixed point α^c of Q_c and bounded by the segments $\overline{(R_{\sigma^i(\hat{\tau}_-)}^c \cup R_{\sigma^i(\hat{\tau}_+)}^c) \cap F^c(1)}$ for $0 < i < q$ union the connecting subarcs of $E^c(1)$. Denote by ι^c the open subarc of $\partial U_0^c \cap E^c(1)$ intersecting the rays $R_{\theta_{\pm}}^c$ and let $\gamma_0^c = \partial U_0^c \setminus \iota^c$. As in the primitive case write $\delta_0^c = \overline{(R_{\theta_-}^c \cup R_{\theta_+}^c) \cap F^c(1)}$ for $c \in \Lambda$.

Then the whole setup is similar to the primitive case. We can thus define $\Omega^c, \Xi_n^c, B_n^c, \hat{\Xi}_{n+1}^c, \hat{B}_{n+1}^c$ for $n \geq 0$ and G^c , all of which moves holomorphically with $c \in \Lambda$. There are however two differences: The first is that the center of $H_{p/q}$ does not belong to Λ . The arguments we used in the primitive case are in-sensitive to a change of base point c_b to another point in the interior of \mathbf{M} . We shall thus take as base point $c_b \in \Lambda$ the center of the central hyperbolic component $H_{p'/q'}^{H_{p/q}} \subset W_{p'/q'}^{H_{p/q}}$. The second difference is that the Yoccoz-Levin parameter inequality is applied to the sublimbs of the hyperbolic component $H_{p'/q'}^{H_{p/q}}$. We leave the details to the reader.

This completes the satellite case and thus completes the proof of Proposition 5.

Proving Theorem 4

As external rays do not cross the proof of Theorem 4 is completely analogous to the proof above of Theorem 1. Let M' be any copy of \mathbf{M} inside \mathbf{M} or \mathbf{M}_1 . In the arguments above replace the carrot field Δ of \mathbf{M} by the dyadic decorations Δ' of M' . Use Yoccoz parameter inequality and the iterated Yoccoz parameter puzzle theorem relative to M' to prove that: $\text{diam}(\Delta'_k) \rightarrow 0$ for any sequence of decorations $(\Delta'_k)_k$ with roots c_k converging to a relatively non renormalizable parameter $c_{\infty} \in M'$. Secondly consider the case $c_{\infty} \in M''$, where $M'' \subset M'$ is a relative to M' first renormalizable copy of \mathbf{M} belonging to some p/q limb of the central hyperbolic component H' of M' . Use again the Yoccoz puzzle theorem relative to M' to define the parameter disk Λ containing M'' similarly as we defined Λ for M' above. And define also Λ^P analogously, i.e. with the aid of the p/q puzzle piece P relative to M' given by Theorem 11 for M' . From here the proof proceeds analogously.

Appendix

In this appendix we supply for completeness proofs of the two theorems we refer to, but for which we have not been able to find either adequate or complete proofs in the literature.

Theorem 7 (The Levin-Yoccoz Dynamical Inequality). *Let H be any hyperbolic component of \mathbf{M} of period k . Let p/q be any non zero reduced rational and*

let $W_{p/q}^H$ denote the relative p/q wake of H , bounded by parameter rays with arguments $0 < \eta_- < \eta_+ < 1$. For any $c \in W_{p/q}^H$ let λ denote the multiplier of the repelling k -periodic common landing point α' of the kq periodic rays $R_{\eta_{\pm}}^c$. Then α' has combinatorial rotation number p/q and λ has a logarithm Λ such that:

$$|\Lambda - p/q2\pi i| \leq \frac{2k \log 2 \cos \theta}{q} \frac{\pi}{\omega(c)},$$

where $\theta \in]-\pi/2, \pi/2[$ is the argument of $\Lambda - p/q2\pi i$ and $\omega(c)$ is the angle of vision of the interval $i2\pi[\eta_-, \eta_+]$ from $\text{Log } \phi_c(c) \in \{z = x + iy | 0 < y < 2\pi\}$.

Proof. Levin proved the fixed point case $k = 1$ in [L, TH. 5.1], the general case is similar. For completeness we sketch here a proof. Let us first recall the proof of Yoccoz inequality (or the Pommerenke-Levin-Yoccoz inequality), full details can be found in [P]. Let T denote the quotient torus $T = D^*/Q_c^k$, where $D^* = \{z | 0 < |z - \alpha'| < r\}$ and $r > 0$ is chosen so small that Q_c^k is univalent on $D = D^* \cup \{\alpha'\}$ and $D \subset\subset Q_c^k(D)$. Let $\Pi : D^* \rightarrow T$ denote the natural projection. The two rays $R_{\eta_{\pm}}^c$ belong to the same orbit and define combinatorial rotation number p/q for α' . Let $\gamma = \Pi(D^* \cap R_{\eta_-}^c) = \Pi(D^* \cap R_{\eta_+}^c)$. Then γ is a Jordan curve and thus the pair (T, γ) has a conformal modulus which satisfies a Grötzsch inequality.

Let $w_{\pm} = \exp(i2\pi\eta_{\pm})$. Then $Q_0^k(w_-) = w_+$ and $Q_c^{kq}(w_{\pm}) = w_{\pm}$. Similarly to T above let \hat{T} denote the quotient torus $\hat{T} = \hat{D}^*/Q_c^k$, where \hat{D}^* is a small punctured disk centered at say w_- and let $\hat{\Pi} : \hat{D}^* \rightarrow \hat{T}$ denote the natural projection. Then $\hat{\Pi}(\hat{D}^* \cap \mathbb{S}^1)$ are two disjoint Jordan curves in \hat{T} , with complement two disjoint, symmetric and straight annuli A_i and A_o . Moreover $\hat{\gamma} = \hat{\Pi}(\hat{D}^* \cap R_{\eta_-}^c)$ is the Jordan equator of A_o and

$$\text{mod}(A_i) + \text{mod}(A_o) = 2\text{mod}(A_o) = \text{mod}(\hat{T}, \hat{\gamma}).$$

If $c \in \mathbf{M}$ so that K_c is connected, the Böttcher coordinate at ∞ induces an isomorphism between A_o and $\Pi(S)$ where S is the connected component of $D^* \cap B^c(\infty)$ containing the end of $R_{\eta_-}^c$. Hence the Grötzsch inequality for (T, γ) implies that

$$\text{mod}(A_o) \leq \text{mod}(T, \gamma). \quad (3)$$

Writting out the values of these two numbers explicitly yields the Yoccoz dynamical inequality: The torus T is isomorphic to $\mathbb{C}^*/\lambda z$ via the linearizer for Q_c^k at α' , or equivalently to $\mathbb{C}/(\mathbb{Z}\Lambda + \mathbb{Z}i2\pi)$ via the log-linearizer. Let $\Pi_u : \mathbb{C} \rightarrow T$ denote the universal covering corresponding to the latter isomorphism. Then the Jordan curve $\gamma = \Pi(D^* \cap R_{\eta_-}^c)$ lifts under Π_u to an arc Γ , which is invariant under the translation $z \mapsto z + L$, where $L = q\Lambda - pi2\pi$ for some appropriate logarithm Λ of λ . A simple computation shows that

$$\text{mod}(T, \gamma) = \frac{2\pi \cos \theta}{q|L|}$$

where θ is the angle between the vector L and the positive real axis. A similar computation shows that

$$2\text{mod}(A_o) = \text{mod}(\widehat{T}, \widehat{\gamma}) = \frac{2\pi}{kq \log 2}.$$

Hence (3) is equivalent to

$$|\Lambda - \frac{p}{q}i2\pi| \leq \frac{2k \log 2 \cos \theta}{q}, \quad (4)$$

which is Yoccoz inequality.

If $c \notin \mathbf{M}$ let $0 \leq \theta < 1$ denote the argument of c , i.e. $c \in R_\theta^c$. Then the Böttcher coordinate ϕ_c at infinity does not extend to a biholomorphic map between $B^c(\infty)$ and $\mathbb{C} \setminus \mathbb{D}$, but almost: It extends to a biholomorphic map of $\mathbb{C} \setminus F^c(h)$ onto $\mathbb{C} \setminus \mathbb{D}(e^h)$ where $h = g_c(c)/2$. Let ψ_c denote the inverse of this extension, then ψ_c extends continuously to $C(0, e^h)$, but this extension is not injective because $0 = \psi_c(\exp(h + 2\pi i\theta/2)) = \psi_c(\exp(h + i2\pi(\theta + 1)/2))$. Let $C = g_c(c) + i2\pi\theta$, $N_0 = [e^{i2\pi\theta}, \phi_c(c)]$ and $N_n = Q_0^{-n}(N_0)$. Define $N_\theta^0 = \cup_{n \geq 0} N_n$ and $N_\theta^1 = \cup_{n \geq 0} N_{n+1}$. Then $Q_0(N_\theta^1) = N_\theta^0$ and ψ_c is easily seen to extend by iterated lifting to a univalent map from $\mathbb{C}_\theta := \mathbb{C} \setminus (\mathbb{D} \cup N_\theta^1)$ into $B^c(\infty)$. The map Q_0 lifts under $\exp(z)$ to the map $z \mapsto 2z$ on \mathbb{C} . That is \exp is a simultaneous linearizer for all the repelling periodic points of Q_0 . The corresponding lifted sets $\tilde{N}_\theta^j = \log(N_\theta^j)$, $j = 0, 1$ are invariant under translation by $i2\pi$ and $2\tilde{N}_\theta^1 = 2\tilde{N}_\theta^0$. Thus if $w = \exp(i2\pi\tau) \in \mathbb{S}^1$ is periodic and if $0 \leq \tau < 1$ does not belong to the orbit of θ , then \mathbb{C}_θ contains a definite sector around the horizontal $\tilde{R}_\tau = \{t + i2\pi\tau | t > 0\}$, which projects to R_τ^0 under \exp : Let $\tau_l < \tau < \tau_r$ be the arguments closest to θ of points in the orbit of w . Then the sectors \tilde{S}_l between $\tilde{R}_{\tau_l} = \{t + i2\pi\tau_l | t > 0\}$ and the oblique line through $i2\pi\tau_l$ in the direction $v_l = C - i2\pi\tau_l$, and \tilde{S}_r between $\tilde{R}_{\tau_r} = \{t + i2\pi\tau_r | t > 0\}$ and the oblique line through $i2\pi\tau_r$ in the direction $v_r = C - i2\pi\tau_r$ are contained in \mathbb{C}_θ : If not some line segment L with $\exp(L) \in N_n$ for some $n \geq 1$ intersects say \tilde{S}_l . But then $2^n L$ intersects the sector $2^n \tilde{S}_l$ with top point $2^n \tau_l$, and is also congruent modulo $i2\pi$ to $L_0 = [i2\pi\theta, C]$ with $\exp(L_0) = N_0$. Since the $2^n \tau_l$ is an argument for a point in the orbit of w this contradicts that τ_l is the closest such argument for points in the orbit of w .

Consequently the sector \tilde{S} around \tilde{R}_τ bounded by the two lines through $i2\pi\tau$ and of directions v_l and v_r is contained in \mathbb{C}_θ .

In the case at hand $c \in W_{p/q}^H$ implies that $\eta_- < \theta < \eta_+$ and for $\eta = \eta_-$ we have $\eta_- = \eta_l$, $\eta_+ = \eta_r$. Let ω_l and ω_r denote the angle of inclination of the vectors $C - i2\pi\tau_l$ and $C - i2\pi\tau_r$ respectively. Then the opening angle ω of \tilde{S} equals $\omega_r - \omega_l$ and the sector \tilde{S} projects to a straight subannulus A_o^θ of A_o with

$$\text{mod}(A_o^\theta) = \frac{\omega}{\pi} \text{mod}(A_o).$$

Arguing as for the proof of the Yoccoz inequality we obtain

$$\text{mod}(A_o^\theta) = \frac{\omega}{\pi} \text{mod}(A_o) \leq \text{mod}(T, \gamma).$$

Properly rewritten as with (4) above, this is the Levin-Yoccoz inequality except for the interpretation of ω . This interpretation is however an elementary exercise in planar geometry and is left to the reader. By continuity the inequality even holds on $\partial W_{p/q}^H$, where either ω_l or ω_r , but not both is zero. \square

Theorem 12. *Let $0 < \eta_- < \eta_+ < 1$ be rationals for which the parameter rays $R_{\eta_{\pm}}^{\mathbf{M}}$ co-land at some point $c_0 \in \mathbf{M}$ and let $W_{\eta_-, \eta_+}^{\mathbf{M}}$ denote the parameter sector bounded by $R_{\eta_-}^{\mathbf{M}} \cup \{c_0\} \cup R_{\eta_+}^{\mathbf{M}}$ and not containing 0. Then the forward orbits of η_{\pm} do not enter the interval $]\eta_-, \eta_+[$. And for any $c \in W_{\eta_-, \eta_+}^{\mathbf{M}}$ the pair of dynamical rays $R_{\eta_{\pm}}^c$ move homomorphically with c , co-land at some repelling (pre)periodic point $z(c)$ with $Q_c^{k'+l}(z(c)) = Q_c^l(z(c))$, where $l \geq 0$ is the common preperiod of η_{\pm} and $k' > 0$ divides the common period $k > 0$ of $\sigma^l(\eta_{\pm})$ and the set $R_{\eta_-}^c \cup \{z(c)\} \cup R_{\eta_+}^c$ bounds a sector W^c containing c , but not 0.*

Proof. This theorem is at least folklore. We supply a proof here for completeness. We shall treat separately the strictly preperiodic case $l > 0$ and the periodic case $l = 0$. For the strictly preperiodic case we have $k = qk'$ with $q > 1$ and c_0 admits precisely q external arguments $0 < \theta_0 < \dots < \theta_{q-1} < 1$ both in dynamical plane and in parameter plane by the Douady-Hubbard ray landing theorem. The arguments $\eta_- < \eta_+$ are amongst these. The set

$$R^c = \bigcup_{i=0}^{q-1} \overline{R_{\theta_i}^c}$$

moves holomorphically with c in $\mathbb{C} \setminus \widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}} = \bigcup_{i=0}^{q-1} \bigcup_{j=1}^{k+l} \overline{R_{\sigma^j(\theta_i)}^{\mathbf{M}}}.$$

Because the Böttcher coordinate ϕ_c depends holomorphically on c and thus R^c moves holomorphically with c as long as c does not belong to the strict forward orbit of R^c . Write $W_{c_0}^{\mathbf{M}}$ for the sector bounded by $R_{\theta_0}^{\mathbf{M}} \cup R_{\theta_{q-1}}^{\mathbf{M}}$. Then $W_{\eta_-, \eta_+}^{\mathbf{M}} \subseteq W_{c_0}^{\mathbf{M}}$ and it suffices to prove that $W_{c_0}^{\mathbf{M}} \cap \widehat{R}^{\mathbf{M}} = \emptyset$. For the later it is enough to prove that

$$\left(\bigcup_{i=0}^{q-1} \bigcup_{j=1}^{k+l} \sigma^j(\theta_i) \right) \cap [\theta_0, \theta_{q-1}] = \emptyset. \quad (5)$$

To this end let us consider the Hubbard tree T^{c_0} for Q_{c_0} . In this strictly preperiodic case T^{c_0} is the minimal connected subset of $K_c = J_c$ containing the orbit

$$\mathcal{O}^{c_0}(0) = \bigcup_{j=0}^{k+l} Q_{c_0}^j(0).$$

As the orbit $\mathcal{O}^{c_0}(0)$ is forward invariant, so is T^{c_0} . Moreover any extremal point of T^{c_0} belongs to $\mathcal{O}^{c_0}(0)$ by minimality. As $Q_{c_0}^j$ is a local homeomorphism for all j the critical value $c_0 = Q_{c_0}(0)$ is necessarily an extremal point. This implies (5). Notice that the conclusion of the theorem holds in this case even for c in a neighbourhood of $\overline{W_{c_0}^{\mathbf{M}}}$.

The periodic case is similar and yet slightly different. The common landing point c_0 of the two parameter rays $R_{\eta_{\pm}}^{\mathbf{M}}$ is the root of a hyperbolic component $H \neq H_0$. Let us rename c_0 to c_1 and use c_0 to denote the center of H . As above the dynamical rays $R_{\eta_{\pm}}^c$ moves holomorphically on $\mathbb{C} \setminus \widehat{R}^{\mathbf{M}}$, where

$$\widehat{R}^{\mathbf{M}} = \bigcup_{j=0}^{k-1} \overline{R_{\sigma^j(\eta_-)}^{\mathbf{M}} \cup R_{\sigma^j(\eta_+)}^{\mathbf{M}}}$$

And to prove the theorem it suffices to prove that $R_{\eta_-}^{c_0}$ and $R_{\eta_+}^{c_0}$ coland at a repelling periodic point $z(c_0)$ in the dynamical plane of Q_{c_0} and that

$$\left(\bigcup_{j=0}^{k-1} \sigma^j(\eta_-) \cup \sigma^j(\eta_+) \right) \cap]\eta_-, \eta_+[= \emptyset.$$

Again the proof is that c_0 is extremal in the Hubbard tree T^{c_0} for Q_{c_0} and that $R_{\eta_{\pm}}^{c_0}$ coland at a k' periodic point $z(c_0)$, $k'|k$ on the boundary of the Fatou component F_0 of c_0 . Notice that in this case the Hubbard tree is defined as the minimal D-H regulated set. Where D-H regulated means that for any Fatou component F the image $\phi(F \cap T^{c_0})$ under the extended Böttcher coordinate consists of radial lines. The proof of extremality of c_0 in T_{c_0} is the same as in the preperiodic case. Also by minimality $z(c_0) = \partial F_0 \cap T^{c_0}$ is the unique periodic point on the boundary of F_0 whose period divides k . By the Douady-Hubbard ray landing theorem $z(c_0)$ is the common landing point of $R_{\eta_{\pm}}^{c_0}$. \square

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